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Analysis of a Combustion Instability Problem Using the Technique of Multiple Scales

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Nomenclature

- a = dimensionless sound speed = $a^*/\bar{a}_i^*$
- B = dimensionless transfer number = $\bar{T} - \tau/\Delta l$
- C = quantity defined after Eq. (1)
- C_p = average specific heat at constant volume of the combustion product gas
- L^* = combustion chamber length
- M = Mach number = $\bar{u}^*/\bar{a}_i^*$
- N = dimensionless droplet number density = $N^*(\rho_l^*/\rho_i^*)(4\pi r_i^{*3}/3)$
- p = dimensionless pressure = $p^*(\bar{p}_i^*)^{1/2}$
- P = dimensionless period of oscillation = $P^*\bar{a}_i^*/L^*$
- r = dimensionless droplet radius = $r^*/\bar{r}_i^*$
- T = dimensionless temperature = $T^*/\bar{T}_i^*$
- t = dimensionless time = $t^*\bar{a}_i^*/L^*$
- u = dimensionless axial velocity = $u^*/\bar{a}_i^*$
- w = dimensionless mass generation = $w^*/\bar{\rho}_{li}^*u_l^*$
- x = dimensionless axial variable = x^*/L^*
- y = dimensionless axial variable = x/η
- γ = ratio of specific heats
- Δl = dimensionless latent heat of vaporization = $\Delta l^*/C_p^*T_i^*$
- δ = dimensionless time variable = $2t/P$
- ϵ = amplitude parameter = $M_e^{1/2}$
- η = dimensionless combustion zone length = η^*/L
- λ = dimensionless thermal conductivity = $\lambda^*/\bar{\lambda}_i^*$
- μ = dimensionless viscosity = $\mu^*/\bar{\mu}_i^*$
- ρ = dimensionless density = $\rho^*/\bar{\rho}_i^*$
- σ = dimensionless entropy difference = σ^*/C_p^*
- τ = dimensionless droplet temperature = $\tau^*/\bar{T}_i^*$

Superscripts

- $^-$ = steady-state quantity
- ^{1} = unsteady quantity
- * = dimensional quantity

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Subscripts

- i = quantity evaluated at the injector face
- l = liquid or droplet variable
- e = quantity evaluated at nozzle entrance

Introduction

THIS Note presents a nonlinear analysis of combustion-driven longitudinal mode pressure oscillations in a liquid rocket combustor.

Mitchell, Crocco, and Sirignano[1] and Crocco and Mitchell[2] have presented nonlinear analyses of similar oscillations in rocket motors having combustion source terms represented by the heuristic sensitive time-lag model devised by Crocco.[3] These analyses relied on the use of the Poincaré-Lighthill strained coordinate technique for their success. The combustion model employed in the work described here is based on a mechanistic droplet vaporization rate controlled model of the type used, for example, by Priem[4] in his instability studies. For this model, in addition to the use of the simplest kind of coordinate stretching, the technique of multiple scale is also employed in order to effectively pursue the analysis of the oscillations. A general discussion of the technique of multiple scales has been given by Cole.[5] It is to be emphasized that the purpose of this presentation is to demonstrate the effectiveness of the technique of multiple scales in the analysis of a particular nonlinear instability problem using a mechanistic combustion model, rather than to present a solution of the problem of nonlinear instability in terms of results with immediate practical application.

Problem Formulation

The combustor under consideration is assumed to be characterized by one-dimensional flow processes and purely axial oscillations. Combustion is assumed to occur between $x = 0$ and $x = \eta$. η is a function of time if oscillations are present. In the combustion region, a two-phase flow consisting of liquid fuel droplets and a constant composition calorically perfect combustion product gas is assumed. No droplet drag forces are considered and, in consequence of this, the liquid droplet velocity u_l is a constant. In addition, the droplet size at injection is taken to be constant at all times. For $x > \eta$ only product gas is present.

The model adopted in order to predict the liquid droplet vaporization rate is the simple collapsed flame model for a droplet in an infinite field combined with an empirical correction term to account for convection. (For a discussion of this model see, for example, Ref. 4). Using this model, the following expression describing the rate of vaporization of the liquid droplets results:

$$D_l r/Dt = -C/r \tag{1}$$

where

$$C = \frac{\lambda_i^* L^*}{C_p^* \rho_L^* a_i^* \bar{r}_i^{*2}} [1 + 0.3Re^{1/3}\mathrm{Pr}^{1/2}] ln(1 + B)$$

$$D_l/Dt = \partial/\partial t + u_l(\partial/\partial x)$$

$$Re = 2r^*\rho^*|u_l^* - u^*|/\mu^*, \quad Pr = C_p^*\mu^*/\lambda^*$$

The conservation equations (mass, momentum, energy) describing the gasdynamic field are written down in the form used by Crocco and Mitchell.[2] The set of three nonlinear partial differential equations in terms of p, u, and σ that results must satisfy a zero velocity condition at the injector and a constant Mach number ("short nozzle," see Ref. 1) condition at the entrance to the nozzle. In addition, all solutions are required to be periodic in time.

Solution of the Equations

The power series approach used to solve the aforementioned set of partial differential equations follows in a general way the technique used previously by Mitchell, Crocco, and Sirignano[1] and Crocco and Mitchell.[2] The relevant dependent variables u, p, r, C, σ and w are represented as power series in ϵ. For example, $p = p_0 + \epsilon p_1 + \epsilon^2 p_2 + \epsilon^3 p_3 + \ldots$ etc., where $\epsilon = M^{1/2}$. The coefficients of ϵ in these expansions are comprised of a steady-state and a time-dependent part. Thus, $p_1 = \bar{p}_1 + p_1^1$, $p_2 = \bar{p}_2 + p_2^1$ etc. Both u_l and $\bar{\eta}$ are taken to be of $0(\epsilon)$. Since, as will be discussed shortly, the analysis is carried out through $0(\epsilon^5)$, the combustion zone can actually occupy a fairly sizeable fraction of the chamber axial length, and is not of zero length as in the "concentrated combustion" model used by Mitchell, Crocco, and Sirignano.[1]

It is necessary to use the simplest kind of coordinate stretching in order to ensure well-behaved periodic solutions even when waveforms are discontinuous. Therefore, the stretched time variable δ is introduced and $P = 2(1 + \epsilon P_1 + \epsilon^2 P_2 + \ldots)$. P is the nondimensional period of the oscillations and 2 is the wave travel time for an acoustic (zero amplitude) wave. The technique of multiple scales is applied with the introduction of a second axial variable y, which is defined as $y = x/\eta$. Derivatives of $\bar{u}$ and $\bar{w}$ with respect to this variable are of $0(1)$ instead of $0(1/\epsilon)$ as they are with respect to x. All dependent variables are then considered to be functions of the 3 independent variables x, y, and δ. Thus, $u = u(x,y,\delta)$, $p = p(x,y,\delta)$, etc. The governing partial differential equations and the droplet vaporization equation are then rewritten in these variables, and the power series representations of the dependent variables are substituted into the equations which result.

The system of equations is first solved for the steady state. To the order of approximation necessary for consistent solution of the time-dependent equations, the results are

$$\bar{u}_2 = \bar{w}_2 = 1 - (1 - y)^{3/2}, \quad \bar{p}_0 = 1, \ \bar{p}_3 = \gamma \bar{u}_2 u_l$$

$$\bar{u}_1 = \bar{w}_1 = \bar{u}_3 = \bar{w}_3 = \bar{p}_1 = \bar{p}_2 = \bar{\sigma}_0 = \bar{\sigma}_1 = \sigma_2 = 0$$

$$r_0 = (1 - y)^{1/2}, \quad \bar{\eta}_1 = u_l/2C_0$$

Carrying out the analysis of the time-dependent equations through $0(\epsilon^3)$ and applying the appropriate boundary conditions leads to the following expressions:

$$u_2^1 = f(\delta - x) - f(\delta + x) \quad p_2^1 = \gamma[f(\delta - x) + f(\delta + x)]$$

$$r_2^1 = \frac{1}{2}(1 - y)^{-1/2}\left[2\int_{\delta}^{\delta - \bar{\psi} y} C_1^1 d\delta^1 - y\,\frac{\eta_2^1}{\bar{\eta}}\right]$$

$$\eta_3^1 = -2\bar{\eta}_1 \int_{\delta - \bar{\psi}}^{\delta} C_1^1 d\delta^1$$

$$u_1^1 = p_1^1 = p_3^1 = u_3^1 = \sigma_1^1 = w_1^1 = w_2^1 = w_3^1 = 0$$

Here, $\bar{\psi} = \bar{\eta}_1/u_l$ (the steady-state droplet lifetime) and f is an arbitrary function periodic in 2. In order to determine the form of f and therefore of p_2 and u_2, it is necessary to continue the analysis through $0(\epsilon^5)$. Doing this and applying the appropriate order boundary conditions, the following nonlinear integro-differential equation for f is finally derived:

$$\left[2P_2 + \sigma_2^1 + (\gamma + 1)f - \frac{3 - \gamma}{2}\langle f \rangle\right]\frac{df}{d\delta} =$$

$$(3\gamma - 1 - 2\gamma d)f + 3\gamma d\int_0^1 (1 - y)^{1/2} f(\delta - \bar{\psi} y)dy \quad (2)$$

where $\sigma_2^1 = 2(\gamma - 1)\langle f \rangle$,

$$\langle f \rangle = \frac{1}{2}\int_{\delta - 2}^{\delta} f(\xi)d\xi, \quad d = \frac{\gamma - 1}{\gamma}\left[\frac{1}{2} + \frac{\bar{B}}{(1 + \bar{B})(1 - \bar{\tau})\ln(1 + \bar{B})}\right]$$

The developments that resulted in Eq. (2) are valid for either continuous or discontinuous pressure waves. In order to determine the waveform and amplitude of the pressure oscillations, this equation must be solved for either type of oscillations. Solution of Eq. (2) for small f is easily carried out by linearizing the equation and assuming $f = \sin\pi\delta$. A neutral stability relationship results. This is

$$d = \frac{3\gamma - 1}{2\gamma}\left[1 - \frac{3}{2}\int_0^1 (1 - y)^{1/2} \cos\bar{\psi} y dy\right] \quad (3)$$

A graph of this equation produces a curve in d, $\bar{\psi}$ space. The region above this curve is linearly unstable, the region below the curve linearly stable. Practical values of $\bar{B}$ and $\bar{\tau}$ cause d to fall somewhat below the lowest point on such a curve. This indicates that the combustion model adopted does not provide a strong enough response to pressure oscillations to drive on instability. In consequence of this, solutions of Eq. (2) for large amplitude waves probably have little practical significance. The conclusion reached is that the combustion model employed is not sufficiently realistic. It is probable that a more sophisticated combustion model would provide a governing equation for f possessing greater practical significance. It is suggested that the use of the multiple scale technique demonstrated here will prove useful in such an analysis (as it certainly was in the present work) in the reduction of the governing system of nonlinear partial differential equations to a single ordinary differential equation determining the nonlinear behavior of the oscillations.

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Numerical Solution of Boundary-Layer Flows with Massive Blowing

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Introduction

DESCRIPTIONS of laminar boundary layers with large mass injection rates are required in several applications. Specifically, of current interest is the stagnation region on the heat shield for a Jupiter atmospheric entry probe.[1] In this case, the mass transfer from the wall due to radiative heating is so great that any convective heating is blocked.

The structure of boundary layers with massive blowing consists of a thin region with large gradients that separates the outer stream from an inner layer adjacent to the wall where gradients are either absent or very small. This boundary-layer structure has previously been examined by several investigators.[2–4]

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